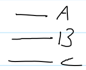
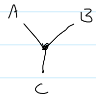
Mr. Feynman and Mr. Lagrangian

Recall the ABC's:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_A\partial^\mu\phi_A - \frac{1}{2}\left(\frac{m_A c}{\hbar}\right)^2\phi_A^2 + \frac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \frac{1}{2}\left(\frac{m_B c}{\hbar}\right)^2\phi_B^2 + \frac{1}{2}\partial_\mu\phi_C\partial^\mu\phi_C - \frac{1}{2}\left(\frac{m_C c}{\hbar}\right)^2\phi_C^2 - g\,\phi_A\phi_B\phi_C$$

In evaluating diagrams built from 

w/ 

we used: $-ig$ vertex factor

$\frac{i}{q_j^2 - m_j^2 c^2}$ virtual particle propagators

There is actually a systematic way to extract the Feynman rules for a given Lagrangian, but to derive it requires QFT. We will simply state the results.

ABC example

Vertex factors: Write $i\mathcal{L}_{int}$ in momentum space ($i\hbar\partial_\mu \to p_\mu$; $\partial_\mu \to -\frac{i}{\hbar}p_\mu$) $\qquad -ig\phi_A\phi_B\phi_C$

Erase the field variables $\qquad -ig$

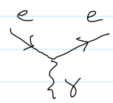
Propagators: Write the relevant free-particle e.o.m. in momentum space and lose overall factor of $\hbar$ $\qquad \partial_\mu\partial^\mu\phi + \left(\frac{mc}{\hbar}\right)^2\phi = 0 \Rightarrow \left[p^2 - (mc)^2\right]\phi = 0$

Multiply the inverse of the term in brackets $\times\, i$: $\qquad \frac{i}{p^2 - m^2c^2}$

Who's house?! Dirac's house!

Consider the QED Lagrangian:

$$\mathcal{L} = i\hbar c\bar\psi\gamma^\mu\partial_\mu\psi - mc^2\bar\psi\psi - q\bar\psi\gamma^\mu\psi A_\mu - \frac{1}{16\pi}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

The fundamental vertex is  or any other charged particle

and the vertex factor we get from $i\mathcal{L}_{int} = -iq\bar\psi\gamma^\mu\psi A_\mu$ removing $\bar\psi, \psi, \overbrace{\sqrt{\frac{\hbar c}{4\pi}}}^{\text{due to units carried by } A_\mu} A_\mu$ giving: $-i\sqrt{\frac{4\pi}{\hbar c}}\,q\gamma^\mu \equiv i g_e\gamma^\mu$

For the propagators we can have both virtual photons and electrons/positrons.

Electrons/Positrons we use: $i\gamma^\mu\partial_\mu\psi - \frac{mc}{\hbar}\psi = 0 \Rightarrow [\gamma^\mu p_\mu - mc]\psi = 0 \Rightarrow \dfrac{i(\gamma^\mu p_\mu + mc)}{p^2 - m^2c^2}$

For photons: Start w/ the massive Proca equation: $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + (\frac{mc}{\hbar})^2 A^\nu = 0$

$$\partial_\mu \to -\tfrac{i}{\hbar}p_\mu \qquad -\tfrac{i}{\hbar}p_\mu\left(-\tfrac{i}{\hbar}p^\mu A^\nu + \tfrac{i}{\hbar}p^\nu A^\mu\right) + \left(\tfrac{mc}{\hbar}\right)^2 A^\nu = 0$$

We want this as one thing operating on A^ν so let's massage it a bit. First get rid of $\hbar$ and i.

$$-p_\mu p^\mu A^\nu + p_\mu p^\nu A^\mu + (mc)^2 A^\nu = 0$$

Multiply by $\eta_{\lambda\nu}$ on both sides.

$$\eta_{\lambda\nu}(-p^2 A^\nu + p_\mu p^\nu A^\mu) + (mc)^2\eta_{\lambda\nu}A^\nu = 0$$

$$\eta_{\lambda\nu}(-p^2 + (mc)^2)A^\nu + \underbrace{p_\mu p_\lambda A^\mu}_{\text{rename } \mu\to\nu \text{ since repeated}} = 0$$

$$[\eta_{\lambda\nu}(-p^2+(mc)^2) + p_\nu p_\lambda]A^\nu = 0$$

Everywhere replace $\lambda \to \mu$

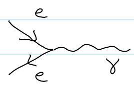
$$[\eta_{\mu\nu}(-p^2+(mc)^2) + p_\mu p_\nu]A^\nu = 0$$

For $m \neq 0$ the propagator is: $i[\;]^{-1} = \overbrace{\dfrac{-i}{p^2-(mc)^2}\left[\eta_{\mu\nu} - \dfrac{p_\mu p_\nu}{(mc)^2}\right]}^{\text{Useful for weak interactions later}}$

For $m=0$ we also need transversality, i.e. $p_\nu A^\nu = 0$ leaving

$$[-\eta_{\mu\nu}p^2]A^\nu \Rightarrow i[\;]^{-1} = \frac{-i\eta_{\mu\nu}^{-1}}{p^2} = \underbrace{\frac{-i\eta_{\mu\nu}}{p^2}}_{\text{Photon propagator}}$$

QED (for now just charged leptons e, μ, τ and photons γ)

Diagrams are built from :  (or μ, τ versions)

In the ABC theory, all d.o.f. were scalars so : a) order was unimportant
b) we were guaranteed M would be a scalar

In QED e, μ, τ are spinors and γ is a vector so: c) order is important
b) we have to be careful to ensure M is a scalar

One new complication is that in most experiments we used unpolarized in-states and sum over all out states.
This means in calculating $|M|^2$ we must a) average over incoming spin states
b) sum over outgoing spin states

Useful expressions:	Electron	Positron	Photon
Wavefunction	$\psi(x) = a e^{-\frac{i}{\hbar} p\cdot x} u^{(s)}(p)$	$\psi(x) = a e^{\frac{i}{\hbar} p\cdot x} v^{(s)}(p)$	$A_\mu(x) = a e^{-\frac{i}{\hbar} p\cdot x} \epsilon_\mu^{(s)}$ $\quad s = 1,2$
E.o.m. (mom. space)	$(\gamma^\mu p_\mu - mc) u = 0$ DE	$(\gamma^\mu p_\mu + mc) v = 0$ DE	$p^\mu \epsilon_\mu = 0$ LC, $\epsilon^0 = 0$ CG
Adjoint	$\bar{u} = u^\dagger \gamma^0$	$\bar{v} = v^\dagger \gamma^0$	$\epsilon^{\mu *}$
Adjoint E.o.m.	$\bar{u}(\gamma^\mu p_\mu - mc) = 0$	$\bar{v}(\gamma^\mu p_\mu + mc) = 0$	$p_\mu \epsilon^{\mu *} = 0$
Orthonormality	$\bar{u}^{(s)} u^{(s')} = 2mc\, \delta_{ss'}$	$\bar{v}^{(s)} v^{(s')} = -2mc\, \delta_{ss'}$	$\epsilon_\mu^{(s)} \epsilon^{\mu * (s')} = -\delta_{ss'}$
Completeness	$\sum_s u^{(s)} \bar{u}^{(s)} = \gamma^\mu p_\mu + mc$	$\sum_s v^{(s)} \bar{v}^{(s)} = \gamma^\mu p_\mu - mc$	$\sum_s \epsilon_i^{(s)} \epsilon_j^{(s)*} = \delta_{ij} - \hat{p}_i \hat{p}_j$

The Dirac spinors for $\vec{p} \neq 0$ in the new convention are:

$$u^{(1)} = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} 1 \\ 0 \\ \frac{c p_z}{E+mc^2} \\ \frac{c(p_x + i p_y)}{E+mc^2} \end{pmatrix} \qquad u^{(2)} = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - i p_y)}{E+mc^2} \\ \frac{-c p_z}{E+mc^2} \end{pmatrix}$$

$$v^{(1)} = \sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} \frac{c(p_x - i p_y)}{E+mc^2} \\ \frac{-c p_z}{E+mc^2} \\ 0 \\ 1 \end{pmatrix} \qquad v^{(2)} = -\sqrt{\frac{E+mc^2}{c}} \begin{pmatrix} \frac{c p_z}{E+mc^2} \\ \frac{c(p_x + i p_y)}{E+mc^2} \\ 1 \\ 0 \end{pmatrix}$$

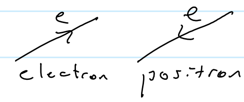
And for the spinor matrices:

$$\gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

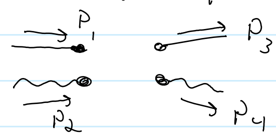
w/ $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Feynman Rules for QED

1. Draw diagram and label matter lines w/ arrows to distinguish particles/anti, e.g.

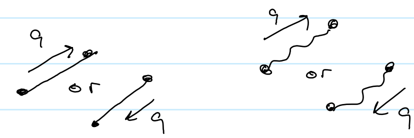


label momenta: a) external

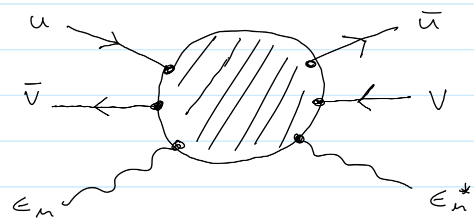


(along T)

b) internal



2. Each <u>external</u> line gets a factor according to:



Number factors w/ momenta, e.g. $\xrightarrow{p_1} \Rightarrow u(1)$

Note: Each factor is a 4spinor or 4vector.

3. Each vertex gets a factor $i g_e \gamma^\mu$ $\left(g_e = e\sqrt{\frac{4\pi}{\hbar c}}\right)$ Note: Overall spin matrix.

4. Each <u>internal</u> line gets a factor: Electrons/Positrons $\dfrac{i(\gamma^\mu q_\mu + mc)}{q^2 - m^2c^2}$ Overall spin matrix.

   Photon $-\dfrac{i\eta_{\mu\nu}}{q^2}$ Overall tensor (μ,ν match vertex indices)

5. Conserve 4momentum at each vertex w/ $(2\pi)^4 \delta^4(p_{in} + q_{in} - p_{out} - q_{out})$

6. Integrate over internal momenta w/ $\int \frac{d^4q}{(2\pi)^4}$ for each q.

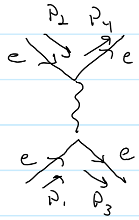
7. Cancel overall $(2\pi)^4 \delta^4(p_{in} - p_{out})$ and $\times i$ to get $\mathcal{M}$.

8. Antisymmetrize between diagrams related by switching 2 incoming electrons/positrons, 2 outgoing electrons/positrons or one incoming electron/positron with one outgoing positron/electron.

9. To get the order right for spinor elements (since we suppress indices) make "spinor sandwhiches from matter lines" by starting w/ an outgoing matter particle line ($\bar{u}$) or incoming antimatter line ($\bar{v}$) and following along <u>only</u> matter segments writing vertex factors and internal matter propagators as we encounter them, until we eventually emerge on an outgoing antimatter (v) or incoming matter (u) line. The photon factors are less tricky since index notation gets them right.

Example:

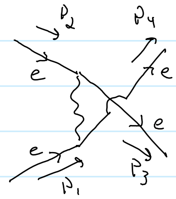
$e + e \to e + e$



$$\int \bar{u}(3)\, i g_e \gamma^\mu\, u(1)\, \bar{u}(4)\, i g_e \gamma^\nu\, u(2) \left(\frac{-i\eta_{\mu\nu}}{q^2}\right) (2\pi)^4 \delta^4(p_1 - q - p_3)\, (2\pi)^4 \delta^4(p_2 + q - p_4) \frac{d^4 q}{(2\pi)^4}$$

$$\bar{u}(3)\, i g_e \gamma^\mu\, u(1)\, \bar{u}(4)\, i g_e \gamma^\nu\, u(2) \left(\frac{-i\eta_{\mu\nu}}{(p_4 - p_2)^2}\right) (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

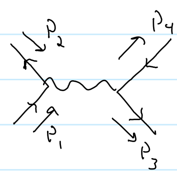
$$\Rightarrow M_1 = \bar{u}(3)\, i g_e \gamma^\mu\, u(1)\, \bar{u}(4)\, i g_e \gamma^\nu\, u(2) \left(\frac{-i\eta_{\mu\nu}}{(p_4 - p_2)^2}\right) i$$



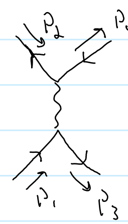
$M_2 = M_1\ (3 \leftrightarrow 4)$

By step 8: $M = M_1 - M_2$

Consider $e^+ + e^- \to e^+ + e^-$:

1. 

M_1

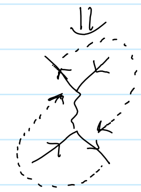
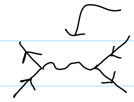
2. 

M_2

$M \stackrel{?}{=} M_1 \pm M_2$

$\Downarrow$

switch one outgoing e^- with one incoming e^+.

 $\Rightarrow$  $=$ 

this is the same as in case 1.

So $M = M_1 - M_2$